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Phase transition of a system of two-level atoms

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Abstract. A very simple treatment is presented to describe a phase transition present in Dicke's model for super-radiance, in the off-resonance case and for a single radiation mode. The results agree with previous authors, although the use of $|n\rangle$ states provides a transparent physical interpretation.

Our results show that if the A^2 term is kept in the Hamiltonian, the phase transition persists, thus clarifying some recent controversy on the subject. The thermodynamic variables are calculated above and below T_c. The phase transition is shown to be a consequence of an instability of the ground state. The behaviour of $\langle a^\dagger a\rangle$ and $\langle S_Z\rangle$ is explored as a function of temperature. $\langle a^\dagger a\rangle$ goes to zero as $(T_c - T)$ near the critical temperature.

1. Introduction

The problem of interaction between a spin system and electromagnetic radiation is the central one in quantum optics. Many advances have been made since Dicke (1954) suggested the idea that atoms could interact in a collective way, via the electromagnetic field. The statistical behaviour of the Dicke model has been extensively studied by various authors (Tavis and Cummings 1968, Scharf 1970, Narducci *et al* 1973a, b).

A considerable interest was generated when a phase transition was found for certain values of the coupling constant (Hepp and Lieb 1973a). When the coupling constant is larger than a critical value, a second-order phase transition between 'normal' and 'super-radiant' states was found. The name 'super-radiant states' was introduced by Hepp and Lieb. It is quite clear that here we are dealing with a thermal equilibrium situation, as opposed to the case of super-radiant pulses, which are fast time-dependent phenomena. The similitude between the two is that in normal and super-radiant pulses, the maximum intensity of the emitted light pulse is proportional to N and N^2 respectively and in the equilibrium case, the average number of photons is proportional to N below T_c and zero above T_c.

A similar condition to that of Hepp and Lieb was found in connection with an instability of the ground state, in the sense that the ground state of the Hamiltonian changes from the vacuum state (no photons, all spins down) to a state with non-zero number of excitations when the coupling constant becomes larger than the critical value (Narducci *et al* 1973a).

Wang and Hioe (1973) were able to derive the phase transition in a simple way, making use of Glauber's coherent states. They calculated the free energy of this model, in the thermodynamic limit.

In this paper, we study the second-order phase transition of Dicke's model, making use of the $|n\rangle$ states. In § 2 the model is briefly reviewed outlining the main features; in § 3, the phase transition is obtained. The $|n\rangle$ states for the field provide a transparent physical interpretation of the results. In § 4 we deal with Dicke's Hamiltonian when the A^2 term, which is normally neglected, is included. The results show that the phase transition is not modified, thus clarifying some recent controversy on the subject (Rzążewski *et al* 1975, Rzążewski and Wódkiewicz 1976, Gilmore and Bowden 1976a, b, Gilmore 1976, Orszag 1977a). Then § 5 is devoted to calculations of the various thermodynamic variables of interest, below and above the critical temperature.

2. The model

Consider a system of N identical two-level atoms, interacting via the electromagnetic field, through dipole interaction, in a cavity of volume V. Assume also that the wavelength is much larger than $(V)^{1/3}$.

Using the rotating-wave approximation, the Hamiltonian is:

$$H = a^\dagger a + \frac{\epsilon}{2}\sum_{i=1}^{N}\sigma_i^z + \frac{\lambda}{2\sqrt{N}}\sum_{j=1}^{N}(a^\dagger\sigma_j^- + a\sigma_j^+), \tag{1}$$

where $a^\dagger$ and a are the usual creation and annihilation operators and σ_i^z is the z component of Pauli spin matrices.

In equation (1), ϵ is the non-resonant parameter. When $\epsilon = 1$, the frequency of the radiation is identical with the frequency corresponding to the two levels. A single radiation mode is considered here; λ is the coupling constant.

3. The phase transition

Following Wang and Hioe, we can write the Hamiltonian (1) in the following form:

$$H = \sum_{j=1}^{N}\left[\frac{a^\dagger}{\sqrt{N}}\frac{a}{\sqrt{N}} + \frac{\epsilon}{2}\sigma_j^z + \frac{\lambda}{2}\left(\frac{a^\dagger}{\sqrt{N}}\sigma_j^- + \frac{a}{\sqrt{N}}\sigma_j^+\right)\right]. \tag{2}$$

The thermodynamics of this model can be calculated, once the partition function is known. The partition function is defined as:

$$Z = \mathrm{Tr}[\exp(-\beta H)]. \tag{3}$$

Using the $|n\rangle$ states for the field, the partition function can be written as:

$$Z = \sum_{S_1=\pm1}\sum_{S_2=\pm1}\cdots\sum_{S_N=\pm1}\sum_{n=0}^{\infty}\left[\!\left[\langle S_1, S_2 \ldots S_N|\langle n|\exp\left\{-\beta\left[\sum_{j=1}^{N}\frac{a^\dagger}{\sqrt{N}}\frac{a}{\sqrt{N}} + \frac{\epsilon}{2}\sigma_j^z + \frac{\lambda}{2}\left(\frac{a^\dagger}{\sqrt{N}}\sigma_j^- + \frac{a}{\sqrt{N}}\sigma_j^+\right)\right]\right\}|n\rangle|S_1 \ldots S_N\rangle\right]\!\right]. \tag{4}$$

In the thermodynamic limit, we will assume:

$$N \to \infty, \qquad V \to \infty, \qquad (N/V) = \text{finite}, \tag{5}$$

and also

$$\left[\frac{a}{\sqrt{N}}, \frac{a^{\dagger}}{\sqrt{N}}\right] = \frac{1}{N} \to 0. \tag{6}$$

The equation (6) was used by Wang and Hioe. A rigorous proof of this method was presented by Hepp and Lieb (1973b).

Assuming equation (6) to be true, in the thermodynamic limit, then $(a^{\dagger}a/N)$ commutes with the rest of the Hamiltonian and we can write:

$$Z = \sum_{S_1=\pm 1} \sum_{S_2=\pm 1} \cdots \sum_{S_N=\pm 1} \sum_{n=0}^{\infty} [\exp(-\beta n)]\langle S_1 \ldots S_N|\langle n| \times \exp\left\{-\beta \sum_{j=1}^{N} \left[\frac{\epsilon}{2}\sigma_j^z + \frac{\lambda}{2}\left(\frac{a^{\dagger}}{\sqrt{N}}\sigma_j^- + \frac{a}{\sqrt{N}}\sigma_j^+\right)\right]\right\}|n\rangle|S_1 \ldots S_N\rangle. \tag{7}$$

Define:

$$h_j = \frac{\epsilon}{2}\sigma_j^z + \frac{\lambda}{2}\left(\frac{a^{\dagger}}{\sqrt{N}}\sigma_j^- + \frac{a}{\sqrt{N}}\sigma_j^+\right), \tag{8}$$

then

$$[h_i, h_j] = 0; \tag{9}$$

therefore:

$$\exp\left(-\beta \sum_{i=1}^{N} h_i\right) = \prod_{i=1}^{N} \exp(-\beta h_i). \tag{10}$$

So we can write the partition function Z as:

$$Z = \sum_{n=0}^{\infty} [\exp(-\beta n)]\langle n|[\langle 1|\exp(-\beta h)|1\rangle + \langle -1|\exp(-\beta h)|-1\rangle]^N|n\rangle \tag{11}$$

where

$$h = \frac{\epsilon}{2}\sigma^z + \frac{\lambda}{2}\left(\frac{a^{\dagger}}{\sqrt{N}}\sigma^- + \frac{a}{\sqrt{N}}\sigma^+\right).$$

It is a simple problem to diagonalise h, namely:

$$\begin{vmatrix} \frac{\epsilon}{2} - \mu & \frac{\lambda a}{\sqrt{N}} \\ \frac{\lambda a^{\dagger}}{\sqrt{N}} & -\frac{\epsilon}{2} - \mu \end{vmatrix} = 0, \tag{12}$$

where μ are the eigenvalues of h. Notice that we will carry out the diagonalisation as if $a/N^{1/2}$ and $a^{\dagger}/N^{1/2}$ were c numbers. Since they commute, a function of this operator can only be interpreted as a power series expansion and the order in which they appear is not important. The solutions for μ are:

$$\mu = \pm\left[\frac{\epsilon^2}{4} + \lambda^2\left(\frac{a^{\dagger}a}{N}\right)\right]^{1/2}. \tag{13}$$

Therefore the partition function can be written as follows:

$$Z=\sum_{n=0}^{\infty}[\exp(-\beta n)]\langle n|\left[\!\left[2\cosh\left\{\frac{\epsilon\beta}{2}\left[1+\frac{4\lambda^2}{\epsilon^2}\left(\frac{a^\dagger a}{N}\right)\right]^{1/2}\right\}\right]\!\right]^N|n\rangle,$$

or

$$Z=\sum_{n=0}^{\infty}[\exp(-\beta n)]\left[\!\left[2\cosh\left\{\frac{\epsilon\beta}{2}\left[1+\frac{4\lambda^2}{\epsilon^2}\left(\frac{n}{N}\right)\right]^{1/2}\right\}\right]\!\right]^N. \tag{14}$$

In a more compact form:

$$Z=\sum_{\substack{x=0\\ [\text{Steps } 1/N]}}^{\infty}\left[\!\left[[\exp(-\beta x)]\left\{2\cosh\left[\frac{\epsilon\beta}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}x\right)^{1/2}\right]\right\}\right]\!\right]^N, \tag{15}$$

where $x=n/N$.

In the thermodynamic limit, only the largest term in the summation will contribute, so:

$$\lim_{N\to\infty}\frac{1}{N}\ln Z=\left[\!\left[-\beta x+\ln\left\{2\cosh\left[\frac{\epsilon\beta}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}x\right)^{1/2}\right]\right\}\right]\!\right]_{x=x_{\max}}, \tag{16}$$

where $x_{\max}$ is the value of x that gives the maximum contribution to Z in equation (15). To find $x_{\max}$, set:

$$\frac{d}{dx}\left[\!\left[-\beta x+\ln\left\{2\cosh\left[\frac{\epsilon\beta}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}x\right)^{1/2}\right]\right\}\right]\!\right]=0. \tag{17}$$

The solution for equation (17) is:

$$\tanh\left(\frac{\epsilon\beta}{2}\eta\right)=\frac{\epsilon}{\lambda^2}\eta, \tag{18}$$

where:

$$\eta=\left(1+\frac{4\lambda^2}{\epsilon^2}x\right)^{1/2}. \tag{19}$$

Obviously, equation (18) is transcendental in x and cannot be solved explicitly. In spite of this fact, the analysis of equation (18) is the key to the study of the phase transition.

Since $0\leqslant x\leqslant\infty$, from equation (19) we see that $1\leqslant\eta\leqslant\infty$; therefore, equation (18) has a solution only if:

$$\lambda^2>\epsilon. \tag{20}$$

In the light of this result, we can identify two regions.

Region 1

$$\lambda<\sqrt{\epsilon}. \tag{21}$$

The largest term in the summation of equation (15) corresponds to $x=0$, therefore

$$Z=\left[2\cosh\left(\frac{\beta\epsilon}{2}\right)\right]^N \tag{22}$$

and there is no phase transition present.

Region 2

$$\lambda > \sqrt{\epsilon}.$$

This case can be studied graphically. From figure 1, we see that:

(*a*) if $\beta < \beta_c$, there is no solution $\eta > 1$; therefore $x_{max} = 0$ and Z is given by equation (22);

(*b*) if $\beta \geqslant \beta_c$ then equation (18) has a solution $x_{max} \neq 0$. From figure 1, we can readily see that the critical temperature is obtained by setting $\eta = 1$ in equation (18), namely:

$$\tanh\left(\frac{\epsilon\beta_c}{2}\right) = \frac{\epsilon}{\lambda^2}. \tag{23}$$

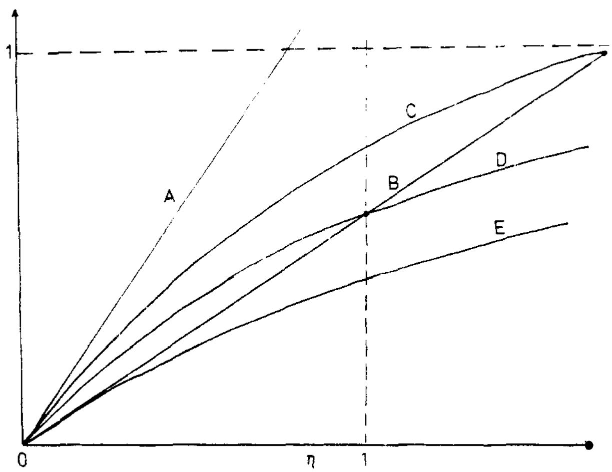

Figure 1. Graphical analysis of the equation $\tanh(\frac{1}{2}\epsilon\beta\eta) = (\epsilon/\lambda^2)\eta$. The straight lines A and B are plots of $(\epsilon/\lambda^2)\eta$ against η for: A, $\lambda^2 < \epsilon$; B, $\lambda^2 > \epsilon$. The curves C, D and E are plots of $\tanh(\frac{1}{2}\epsilon\beta\eta)$ against η for: C, $\beta > \beta_c$; D, $\beta = \beta_c$; E, $\beta < \beta_c$. In region 1 $(\epsilon/\lambda^2 > 1)$, there is no solution; in region 2 $(\epsilon/\lambda^2 < 1)$, there is a solution for $\eta > 1$ provided $\beta > \beta_c$.

Summarising the results of this section, the Dicke model can describe a phase transition if there is a strong coupling between atoms and the field. The critical coupling constant is modified if counter-rotating terms are included in the Hamiltonian (Carmichael *et al* 1973, Orszag 1977a).

4. The controversy

As mentioned in the introduction, a controversy was originated when Rzążewski, Wódkiewicz and Żakowicz suggested that the presence of the phase transition was due entirely to the absence of the A^2 term in the Hamiltonian. The countervailing viewpoint was supported by Gilmore, Bowden and Orszag (Gilmore and Bowden 1976a, b, Gilmore 1976, Orszag 1977a, b).

In this section, we include the A^2 term in the Hamiltonian and determine its effect on the phase transition.

The modified Hamiltonian will now be:

$$H=\sum_{j=1}^{N}\left[\frac{a^\dagger}{\sqrt{N}}\frac{a}{\sqrt{N}}+\frac{\epsilon}{2}\sigma_j^z+\frac{\lambda}{2}\left(\frac{a^\dagger}{\sqrt{N}}\sigma_j^-+\frac{a}{\sqrt{N}}\sigma_j^+\right)+K\left(\frac{a}{\sqrt{N}}+\frac{a^\dagger}{\sqrt{N}}\right)^2\right]. \quad (24)$$

where $a^\dagger a/N$ as well as $(a^\dagger/N^{1/2}+a/N^{1/2})^2$ commute with the Hamiltonian. The partition function Z is:

$$Z=\sum_{S_1=\pm1}\sum_{S_2=\pm1}\cdots\sum_{S_N=\pm1}\sum_{n=0}^{\infty}[\exp(-\beta n)]\langle S_1\ldots S_N|\langle n|\prod_{j=1}^{N}\exp\left(-\frac{\beta K}{N}(a+a^\dagger)^2\right)$$
$$\times\exp\left\{-\beta\left[\frac{\epsilon}{2}\sigma_j^z+\frac{\lambda}{2}\left(\frac{a^\dagger}{\sqrt{N}}\sigma_j^-+\frac{a}{\sqrt{N}}\sigma_j^+\right)\right]\right\}|n\rangle|S_1\ldots S_N\rangle. \quad (25)$$

Summing over the spin variables, we get:

$$Z=\sum_{n=0}^{\infty}[\exp(-\beta n)]\langle n|\left[\!\left[\exp\left(-\frac{\beta K}{N}(a+a^\dagger)^2\right)\left\{2\cosh\left[\frac{\epsilon\beta}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}\frac{a^\dagger a}{N}\right)^{1/2}\right]\right\}\right]\!\right]^N|n\rangle \quad (26)$$

or simply:

$$Z=\sum_{n=0}^{\infty}[\exp(-\beta n)]\left\{2\cosh\left[\frac{\epsilon\beta}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}\frac{n}{N}\right)^{1/2}\right]\right\}^N\langle n|\exp[-\beta K(a+a^\dagger)^2]|n\rangle. \quad (27)$$

It is simple to prove that (see appendix):

$$\langle n|\exp[-\beta K(a+a^\dagger)^2]|n\rangle=[\exp(-2\beta Kn)]I_0(2\beta Kn), \quad (28)$$

where $I_0(2\beta Kn)$ is the zeroth-order modified Bessel function. We write the partition function as:

$$Z=\sum_{\substack{x=0\\ \text{[steps } 1/N]}}^{\infty}\exp\left[\!\left[-Nx\beta+N\ln\left\{2\cosh\left[\frac{\epsilon\beta}{2}\left(1+\frac{4\lambda^2}{\epsilon^2}x\right)^{1/2}\right]\right\}\right.\right.$$
$$\left.\left.-2N\beta Kx+\ln(I_0(2\beta KxN))\right]\!\right]. \quad (29)$$

If the maximum corresponds to $x_{\max}=0$, then the last two terms of equation (29) vanish, and there is no contribution from the A^2 term. On the other hand, if the maximum corresponds to $x_{\max}\neq 0$, then when $N\to\infty$,

$$I_0(2\beta KxN)\to[\exp(2\beta KxN)]/[(2\pi)(2\beta KxN)]^{1/2}. \quad (30)$$

The last two terms of equation (29) corresponding to the A^2 contribution, will be:

$$-2N\beta Kx+\ln(I_0(2\beta KxN))=-\tfrac{1}{2}\ln(4\pi\beta KxN).$$

As a conclusion of this section, we state that the A^2 term in Dicke's Hamiltonian does not affect the phase transition, since in the thermodynamic limit these terms contribute $(\ln N)/N\to 0$ to the free energy.

Gilmore and Bowden (1976b) arrived at the same conclusion, using a completely different procedure. They used Bogoliubov's method to linearise the Hamiltonian when the A^2 term was included. Then they proceeded to minimise the free energy with respect to the parameters of the transformation. Their Hamiltonian, which is a linearised version of Dicke's, proved to be thermodynamically equivalent to Dicke's Hamiltonian. Therefore it is only meaningful to compare the two methods strictly in the thermodynamic limit. When $N \to \infty$, both methods reach the same conclusion.

5. Thermodynamics

So far, we have not said anything about the nature of the phase transition. In this section we are going to explore the properties of the thermodynamic quantities of interest, across the critical point.

5.1. *Heat capacity*

Define:

$$c(\text{per molecule}) = \frac{1}{k_B T^2} \frac{\partial^2}{\partial \beta^2}\left(\frac{\ln Z}{N}\right). \tag{31}$$

After some algebraic work, the heat capacity is found to be:

$$\begin{aligned} c(\beta_c^-) &= (k_B \beta_c^2 \epsilon^2/4)[(\lambda^4 - \epsilon^2)/\lambda^4] \\ c(\beta_c^+) &= (k_B \beta_c^2 \epsilon^2/4)[(\lambda^4 - \epsilon^2)/(\lambda^4 - \tfrac{1}{2}\beta\lambda^6 + \tfrac{1}{2}\beta\epsilon^2\lambda^2)]. \end{aligned} \tag{32}$$

These results agree with Hepp and Lieb.

5.2. *Magnetisation*

Consider a Hamiltonian of the form:

$$H = H'(\bar{\mu}/2) \sum_{i=1}^{N} \sigma_i^z + H_0,$$

where $\bar{\mu} \equiv g\mu_B$, g being the Landé factor, μ_B Bohr's magneton and H' the external magnetic field.

The magnetisation is defined as:

$$M(T, H') = \frac{1}{Z}\mathrm{Tr}[\bar{\mu} S_Z \exp(-\beta H)], \tag{33}$$

where:

$$S_Z = \sum_{i=1}^{N} \tfrac{1}{2}\sigma_i^z. \tag{34}$$

We can write equation (33) simply as:

$$M = \bar{\mu}\langle S_Z \rangle. \tag{35}$$

In terms of Z the magnetisation becomes:

$$m(T, H') = -\frac{1}{\beta}\frac{\partial}{\partial H'}\left(\frac{\ln Z}{N}\right) \quad \text{(per molecule)}. \tag{36}$$

In the Dicke Hamiltonian, we associate

$$\epsilon \to \bar{\mu} H'$$

so:

$$m_{\text{Dicke}}(T, \epsilon) = -\frac{\bar{\mu}}{\epsilon} \frac{\partial}{\partial \epsilon}\left(\frac{\ln Z}{N}\right). \tag{37}$$

Using equation (29) and the condition stated in equation (23) for $\beta < \beta_c$ and $\beta > \beta_c$, it is simple to show that:

$$m = \begin{cases} -\dfrac{\bar{\mu}}{2} \tanh\left(\dfrac{\epsilon\beta}{2}\right) & \beta < \beta_c, \\ -\dfrac{\bar{\mu}}{2}\left(\dfrac{\epsilon}{\lambda^2}\right) & \beta > \beta_c. \end{cases} \tag{38}$$

Notice that m is a continuous function of T across T_c. This result is in agreement with Gibberd's work.

5.3. *Number of photons*

The quantity $\langle a^\dagger a \rangle$ is of physical interest. We calculate it as follows:

$$\left\langle \frac{a^\dagger a}{N} \right\rangle = \left[-\frac{1}{\beta} \frac{\partial}{\partial \delta}\left(\frac{\ln Z}{N}\right)\right]_{\delta = 1}, \tag{39}$$

where δ is a dummy variable defined by:

$$Z = \sum_{x=0}^{\infty} \left[[\exp(-\beta \delta x)] 2 \cosh\left(\frac{\epsilon\beta}{2} \eta\right)\right]^N.$$

From equation (39) and the partition function, we get:

$$\langle a^\dagger a \rangle = N x_{\max}. \tag{40}$$

There are two cases.

Case 1. $T > T_c$ and $\langle a^\dagger a \rangle = 0$.
Case 2. $T < T_c$ and $\langle a^\dagger a \rangle = N x_{\max}$, where $x_{\max}$ is the solution of equation (18).

6. Discussion

We are now going to analyse briefly the results of the last section. From equation (37), we infer that:

$$\langle S_Z \rangle_{\lambda^2 > \epsilon} = \begin{cases} -\frac{1}{2} N \tanh(\frac{1}{2}\epsilon\beta) & \beta < \beta_c \\ -\frac{1}{2} N(\epsilon/\lambda^2) & \beta > \beta_c \end{cases}$$

$$\langle S_Z \rangle_{\lambda^2 < \epsilon} = -\tfrac{1}{2} N \tanh(\tfrac{1}{2}\epsilon\beta). \tag{41}$$

These results are displayed graphically in figure 2. (An equivalent graph was obtained by Gibberd (1974).)

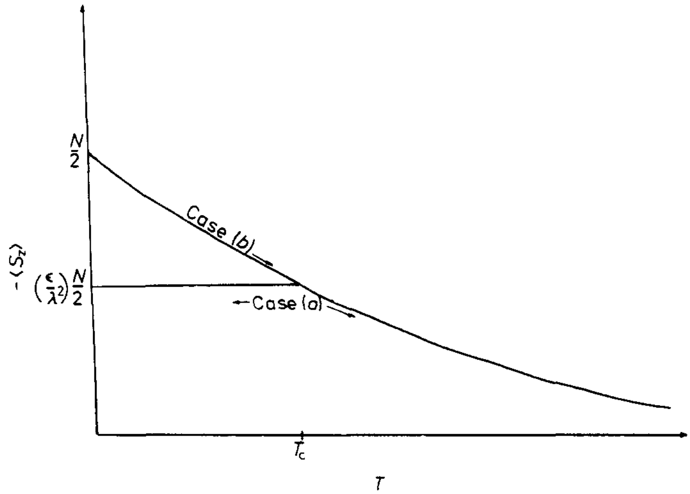

Figure 2. Behaviour of $\langle S_Z \rangle$ as a function of temperature. Case (a) is for $\lambda^2 > \epsilon$ and case (b) for $\lambda^2 < \epsilon$.

At $T=0$, the ground state of the system corresponds to $\langle S_Z \rangle = -\frac{1}{2}N$ when $\lambda < \sqrt{\epsilon}$, which is the state with all spins down. A dramatic change occurs when $\lambda > \sqrt{\epsilon}$, namely that at $T=0$, the ground state corresponds to $\langle S_Z \rangle = -N\epsilon/2\lambda^2$, which indicates the presence of a non-zero number of excitations (Narducci *et al* 1973a). We also notice that $\partial M/\partial T$, $\partial M/\partial \epsilon$, and the heat capacity are discontinuous across T_c. These facts indicate that we are dealing with a second-order phase transition.

Turning now to the average number of photons, at $T=0$ we have:

$$\frac{\epsilon}{\lambda^2}\left(1+\frac{4\lambda^2}{\epsilon^2}x_{T=0}\right)=1, \tag{42}$$

or

$$\langle a^\dagger a \rangle_{T=0} = \frac{N\epsilon^2}{4\lambda^2}\left(\frac{\lambda^4}{\epsilon^2}-1\right), \tag{43}$$

and

$$\left|\frac{\mathrm{d}\langle a^\dagger a\rangle}{\mathrm{d}T}\right|_{T=0} = 0. \tag{44}$$

For $T \to T_c$

$$|\langle a^\dagger a\rangle|_{T\to T_c} \propto (T_c - T). \tag{45}$$

These results are displayed in figure 3. (The square root of this figure was published by Gilmore and Bowden 1976a).

Although this model is semiclassical in the sense that we have neglected the commutation rules for the field, an order parameter has to be defined in order to compare this model with others (Gilmore and Bowden 1976a, b, Stanley 1971).

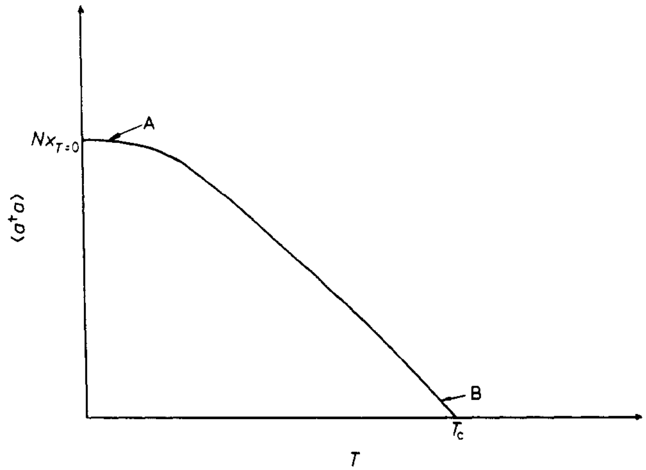

Figure 3. Behaviour of the average number of photons as a function of temperature for $\lambda^2>\epsilon$. If $\lambda^2<\epsilon$, then $\langle a^{\dagger}a\rangle$ is zero for all temperatures. A, $\mathrm{d}\langle a^{\dagger}a\rangle/\mathrm{d}T=0$; B, $\langle a^{\dagger}a\rangle\propto(T-T_c)$.

The coupling constant λ is defined as:

$$\lambda=d\surd(2\pi\rho)/\surd(\hbar\omega)$$

where d is the dipole matrix element.

For a low density gas ($p\sim1$ Torr) at room temperature and optical or higher frequencies, λ is normally much less than unity, and the effect described here is not observable. On the other hand, at the lower part of the spectrum (infrared or microwaves) and higher densities, λ could be unity or higher. Notice that if the density of the radiating medium is increased up to a point where the spatial components of the wavefunctions of the individual particles start to overlap, then we have to symmetrise or anti-symmetrise them, since they are indistinguishable. We have assumed throughout this work that the particles are independent and coupled only through a common field. We have also neglected any direct interaction (collisions, etc). If we take an example of a resonant system, and $\epsilon/\lambda^2=0{\cdot}9$, then the critical temperatures will be $T_c=8160$ K ($\lambda_0=6000$ Å), $T_c=490$ K ($\lambda_0=10\ \mu$m) and $T_c=49$ K ($\lambda_0=100\ \mu$m). A more realistic Hamiltonian, including phonons and with less stringent conditions on the coupling constant is described elsewhere (Orszag 1977b).

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Appendix

Proof of the equation:

$$\langle n|\exp[-\beta K(a+a^{\dagger})^2]|n\rangle=[\exp(-2\beta Kn)]I_0(2\beta Kn).$$

Write:

$$\exp\left[-\beta K\left(\frac{a}{\sqrt{N}}+\frac{a^\dagger}{\sqrt{N}}\right)^2 N\right]$$
$$=\left\{\exp\left[-\beta K\left(\frac{a}{\sqrt{N}}\right)^2 N\right]\right\}\left\{\exp\left[-\beta KN\left(\frac{2a^\dagger a}{N}\right)\right]\right\}\left\{\exp\left[-\beta KN\left(\frac{a^\dagger}{\sqrt{N}}\right)^2\right]\right\} \tag{A.1}$$

and since

$$[a/\sqrt{N}, a^\dagger/\sqrt{N}]=0,$$

we can rewrite equation (A.1) as follows:

$$\exp[-\beta K(a+a^\dagger)^2]$$
$$=\{\exp[-2\beta K(a^\dagger a)]\}\left(1-\frac{\beta K}{1!}(a)^2+\frac{(\beta K)^2}{2!}(a)^4\dots\right)$$
$$\times\left(1-\frac{\beta K}{1!}(a^\dagger)^2+\frac{(\beta K)^2}{2!}(a^\dagger)^4\dots\right). \tag{A.2}$$

Considering that $\exp[-2\beta(a^\dagger a)K]$ contains only diagonal terms, the only surviving terms in (A.2) will be the ones with equal power in a and $a^\dagger$, namely:

$$\{\exp[-2\beta K(a^\dagger a)]\}\left(1+\frac{[\beta K(a^\dagger a)]^2}{(1!)^2}+\frac{[\beta K(a^\dagger a)]^4}{(2!)^2}+\dots\right). \tag{A.3}$$

Using equation (A.3), we finally write:

$$\langle n|\exp[-\beta K(a+a^\dagger)^2]|n\rangle$$
$$=[\exp(-2\beta Kn)]\left(1+\frac{(\beta Kn)^2}{(1!)^2}+\frac{(\beta Kn)^4}{(2!)^2}+\dots\right)$$
$$=[\exp(-2\beta Kn)]I_0(2\beta Kn). \tag{A.4}$$

References

Carmichael H J, Gardiner C W and Walls D F 1973 *Phys. Lett.* **46A** 47
Dicke R H 1954 *Phys. Rev.* **93** 99
Gibberd R W 1974 *Aust. J. Phys.* **27** 241
Gilmore R 1976 *Phys. Lett.* **55A** 459
Gilmore R and Bowden C M 1976a *Phys. Rev.* A **13** 1898
—— 1976b *J. Math. Phys.* **17** 1617
Hepp K and Lieb E H 1973a *Ann. Phys., NY* **76** 360
—— 1973b *Phys. Rev.* A **8** 2517
Narducci L M, Orszag M and Tuft R A 1973a *Collective Phenomena* vol. 1 (New York: Academic) p 113
—— 1973b *Phys. Rev* A **8** 1892
Orszag M 1977a *J. Phys. A: Math. Gen.* **10** L21
—— 1977b *J. Phys. A: Math. Gen.* **10** L25
Rzążewski K and Wódkiewicz K 1976 *Phys. Rev.* A **13** 1967
Rzążewski K, Wódkiewicz K and Żakowicz W 1975 *Phys. Rev. Lett* **35** 432
Scharf G 1970 *Helv. Phys. Acta* **43** 806
Stanley H E 1971 *Introduction to Phase Transition and Critical Phenomena* (London: Oxford University Press)
Tavis M and Cummings F W 1968 *Phys. Rev.* **170** 379
Wang Y K and Hioe F T 1973 *Phys. Rev.* A **7** 831